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Toda field theory as a clue to the geometry of W_n -gravity¹

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Abstract

We discuss geometrical aspects of Toda Fields generalizing the links between Liouville gravity and uniformization of Riemann surfaces of genus greater than one. The framework is the interplay between the hermitian and the holomorphic geometry of vector bundles on such Riemann surfaces.

Pointing out how Toda fields can be considered as equivalent to Higgs systems, we show how the theory of Variations of Hodge Structures enters the game inducing local holomorphic embeddings of Riemann surfaces into homogeneous spaces. The relations of such constructions with previous realizations of W_n -geometries are briefly discussed.

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1 Introduction

As it was put forward by Polyakov [21] in his seminal paper on the geometry of strings, two dimensional euclidean quantum gravity in the conformal gauge is described by Liouville theory. Indeed a fruitful approach to 2D quantum gravity is to look at it as an effective theory resulting from minimally coupling conformal matter $S_{mat}^0(X)$ to metric g on the two-dimensional universe Σ .

Integrating out the matter degrees of freedom in the Feynman path integral, and factoring out the invariance group $Diff(\Sigma)$ of diffeomorphisms of Σ gives rise to an effective action containing the reparametrization ghosts and the conformal factor in the metric $g = e^{2\phi}\hat{g}$.

In such a Faddeev–Popov reduction procedure the field ϕ can be shown to enter the theory through the Liouville action

$$S_{Liouv}(\phi; \hat{g}) \sim \int_{\Sigma} dx dt \sqrt{\hat{g}} (1/2 g^{ab} \partial^a \phi \partial^b \phi + \hat{R} \phi + e^{2\phi}) \quad (1.1)$$

provided some reasonable locality assumptions are made [7] when one takes into account the dependence on the moduli of Σ .

Following the introduction, originally due to A. B. Zamolodchikov [27], of higher spin extensions of the Virasoro algebra in conformal field theory, a generalization of ordinary 2D-gravity, called W_n -gravity has recently received a great deal of attention, also in the attempt of breaking the $c = 1$ barrier characteristic of Liouville theory.

This was both a natural extension of Polyakov’s results about the interpretation of 2D-gravity as a *constrained* $SL(2, \mathbb{R})$ WZNW-theory [22], and of the results about matrix models of discretized gravity, showing their equivalence with the n-KdV equations, whose symmetry algebra is just the (classical limit of the) W_n -algebra.

Some interesting results in W_n -gravity both in the Feynman path integral approach and in the BRST formalism have already been obtained [6, 28, 16, 10, 19]. In this paper we want to investigate on the geometrical properties of Toda systems and discuss what are the “chiral” embeddings of Σ one can associate to a Toda Field. The main tools are the introduction of some auxiliary vector bundles over Σ (i.e. making a detour through gauge theory) and the study of the interplay of their differential and holomorphic geometry. We will stick mainly to the A_n Toda case, and consider compact Riemann surfaces of genus greater than one, a first and easier step in the study of the more general case of negatively curved pointed surfaces.

Actually, as we will clarify in section (2), we follow a path suggested by Liouville theory, and argue that as Liouville theory is the classical Poincaré – Koebe – Klein uniformization of Riemann surfaces, the geometrical structure behind W_n -gravity is the “higher-order uniformization” of Hitchin, Simpson and others, i.e. the geometry of Higgs bundles over Σ , structures that were first introduced by Hitchin [17] in the framework of the self-dual Yang Mills equations, whose rôle in this topics is currently under deep investigation [6, 16].

The starting point is the fact that the Toda equations can be given the form of zero-curvature equations for a suitable connection (the *Toda connection*), and that there is a gauge in which they are equivalent to Hitchin's equation for the corresponding Higgs pair. Making use of the theory of harmonic Higgs bundles [5, 25] one can decompose the Toda connection in a *metric* part plus a deformation α . Then the metric gives rise to *harmonic* local maps from Σ to a symmetric Riemannian manifold, which however, is not enough for our purposes since, in general, the target is *not even a complex manifold*. The analysis can be refined under the light of the theory of Variations of Hodge Structures, thanks to the fact that the Toda connection (and so the associated Higgs system) is quite "special" and satisfies the so-called Griffiths transversality conditions [15, 24]. It follows that one can associate to any solution of the Toda equations a holomorphic map of Σ into a locally homogeneous hermitian manifold.

Recalling the intimate relations [4] between Toda theory and W_n -algebras, a link between such results and previous realizations of W_n geometry [23, 14] will be provided by recovering the generalized Plücker embeddings associated to the A_n Toda systems.

This paper heavily relies on [2], to which we refer for missing proofs, a proper mathematical setting and a more substantial list of references.

2 Some geometrical aspects of Liouville theory

In this section we will recall some facts about the Liouville equations with the aim of casting it in a form suitable to be generalized to that A_n Toda case.

The geometry underlying the Liouville equation is the classical geometry of uniformization. The Liouville equations obtained from the action (1.1) are, in complex coordinates $z = x + \sqrt{-1}t$, $\bar{z} = x - \sqrt{-1}t$,

$$\partial\bar{\partial}\phi = e^{2\phi} \quad (2.1)$$

They are consistent if and only if the Liouville mode ϕ is regarded as the conformal factor in a metric g over Σ

$$g_{z\bar{z}} = e^{2\phi} dz \otimes d\bar{z} \quad (2.2)$$

and can be expressed as the condition for the constancy of the curvature scalar of g .

The universal Riemann surface supporting its standard solution is the complex upper half plane \mathbb{H} endowed with the Poincaré metric

$$ds^2 = e^{2\phi} dz \otimes d\bar{z} = \frac{1}{(\text{Im}z)^2} dz \otimes d\bar{z} \quad (2.3)$$

Since this is invariant under projective transformations

$$z \rightarrow \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \quad (2.4)$$

one can start by finding solutions (i.e. defined in an open simply connected coordinate patch of Σ) by pull-back as

$$e^{2\phi} dz \otimes d\bar{z} = \frac{\partial A \bar{\partial} B}{(A - B)^2} dz \otimes d\bar{z} \quad (2.5)$$

with $\bar{\partial} A = \partial B = 0$. Then reality $\overline{e^{2\phi}} = e^{2\phi}$ is enforced by requiring that $B(\bar{z})$ be at most a real projective transformation of $A(z)$, thereby inducing a real structure. We shall see real structures play a major role in the sequel.

To proceed further, let us define

$$T(z) = e^\phi \partial^2 e^{-\phi} \quad (2.6)$$

$$\bar{T}(\bar{z}) = e^\phi \bar{\partial}^2 e^{-\phi} \quad (2.7)$$

Then, with the usual notation of $\{f, z\}$ for the Schwarzian derivative of f with respect to z , $T = -1/2\{A, z\}$ and $\bar{T} = -1/2\{B, \bar{z}\}$, so that by putting $\xi_1 = \frac{1}{\sqrt{A}}$, $\xi_2 = \frac{A}{\sqrt{A}}$, it is easy to check that $\xi_{1,2}$ are a basis of the space of solutions of the equation

$$-\partial^2 \xi + T(z)\xi = 0 \quad (2.8)$$

To globalize such local solutions to a non-trivial Riemann surface Σ we consider the local system $\Xi_\alpha = (\xi_{1\alpha}, \xi_{2\alpha})$, which solves (2.8) in the open patch U_α , glued in $U_\alpha \cap U_\beta$ with Ξ_β by means of

$$\Xi_\alpha^i = (k_{\alpha\beta})^{-1/2} [S_{\alpha\beta}]_j^i \Xi_\beta \quad (2.9)$$

where $k_{\alpha\beta} = \left(\frac{dz_\beta}{dz_\alpha}\right)$ is the \mathbb{C}^* -cocycle defining the canonical bundle and $[S_{\alpha\beta}]$ is a *flat* $SL(2, \mathbb{C})$ cocycle. Associated to such a bundle $K^{-1/2} \otimes S$ we can consider the jet bundle of $-1/2$ -differentials. This arises in the following way.

Let us differentiate the relation (2.9) with respect to z_β . We get

$$\left(\frac{d}{dz_\beta} k_{\alpha\beta}^{1/2}\right) \Xi_\alpha(z_\alpha) + (k_{\alpha\beta})^{-1/2} \frac{d\Xi_\alpha}{dz_\alpha} = [S_{\alpha\beta}] \frac{d\Xi_\beta}{dz_\beta} \quad (2.10)$$

a relation which shows that the one-cochain $F_\alpha = (\Xi'_\alpha, \Xi_\alpha)$ intertwines between the flat cocycle $S_{\alpha\beta}$ and the cocycle

$$J_{\alpha\beta}^1 = \begin{pmatrix} k_{\alpha\beta}^{1/2} & 0 \\ \partial_\beta \log k_{\alpha\beta} & k_{\alpha\beta}^{-1/2} \end{pmatrix} \quad (2.11)$$

An extensive analysis of such a representation for Liouville theory has been performed in [1]. What is relevant for us is the following picture.

Let us consider the (C^∞) vector bundle $E = K^{-1/2} \oplus K^{1/2}$; a result which has long been known in the literature [20] is that the Liouville equations are the zero-curvature conditions for the connection ∇_A on E defined by

$$d'_A = \partial + \begin{pmatrix} \partial\phi & 1 \\ 0 & -\partial\phi \end{pmatrix}, \quad d''_A = \bar{\partial} + \begin{pmatrix} 0 & 0 \\ e^{2\phi} & 0 \end{pmatrix} \quad (2.12)$$

For our purpose, it is crucial to analyze further this structure and point out some facts. First of all, the term $e^{2\phi}$ appearing in the lower left corner of the connection matrix $A_{\bar{z}}$ has the right covariance properties to be interpreted as a metric g on Σ , since it is a section of $Hom(K^{-1/2}, K^{1/2}) \otimes \bar{K} \simeq K \otimes \bar{K}$, or in other words is a $(1, 1)$ -form; also

$$\omega = \partial\phi = 1/2\partial\log g \quad (2.13)$$

is the *metric* connection on $K^{-1/2}$ relative to the fiber metric \sqrt{g} (and $-\partial\phi$ is the fiber metric on $K^{1/2}$).

Furthermore, the holomorphic fiber bundle supporting the connection ∇_A is the bundle

$$0 \rightarrow K^{1/2} \rightarrow E_2 \rightarrow K^{-1/2} \rightarrow 0 \quad (2.14)$$

of 1-jets of $-1/2$ -holomorphic differentials whose representing cocycle is given by (2.11), and under the gauge transformation $\gamma = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$ the connection ∇_A is transformed into the *analytic* one ∇_W which reads

$$d'_W = \partial + \begin{pmatrix} 1 & 0 \\ -\partial\omega + \omega^2 & 1 \end{pmatrix}, \quad d''_W = \bar{\partial} \quad (2.15)$$

A couple of remarks are in order: from the physical point of view, one can notice that since $\omega = \partial\phi$, we have recovered the improved energy momentum tensor $T = (\partial\phi)^2 - del^2\phi$ as the only non-vanishing component of the analytic connection in the Drinfel'd-Sokolov (or W)-form.

From the mathematical point of view, there is no contradiction about the existence of an analytic flat connection on E_2 , since it is a flat irreducible holomorphic bundle (and, moreover, the unique rank 2 bundle admitting a filtration as in (2.14)), as it can be seen that its extension class is $1/2$ of the Chern class of Σ which is non vanishing as long as the curve is hyperbolic.

The observation which brings Higgs systems into the game is that the Toda connection (2.12) can be interpreted in the form $\nabla_A = \nabla_B + \theta + \theta^*$ where

$$\nabla_B = \partial + \begin{pmatrix} \partial\phi & 0 \\ 0 & -\partial\phi \end{pmatrix} + \bar{\partial} \quad (2.16)$$

is the metric connection on V with respect to the fiber metric $h = \begin{pmatrix} \sqrt{g} & 0 \\ 0 & 1/\sqrt{g} \end{pmatrix}$ (recall that $g = e^{2\phi}$, $\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz$ is a $(1, 0)$ -form with values in $End(V)$ and θ^* is the *metric adjoint* of θ). The zero curvature (alias Toda) equations $F(A) = 0$ are then traded for the equations

$$F(B) = -[\theta, \theta^*] \quad (2.17)$$

$$\bar{\partial}\theta = 0 \quad (2.18)$$

stating that actually θ is holomorphic, and that (E, θ) constitute a stable Higgs pair [17]. This will be the form of Liouville theory we are going to discuss in the n -component Toda case.

3 Higgs bundles, Harmonic bundles, and the Toda equations

Let Σ be a genus g Riemann surface with canonical bundle K . A Higgs bundle over Σ is a pair (E, θ) with E a holomorphic rank n vector bundle over Σ and θ a holomorphic $\text{End}(E)$ -valued $(1, 0)$ form. A Higgs bundle is stable if the slope (i.e. the ratio of the first Chern class with the rank) of every non-trivial θ -invariant subbundle $F \subset E$ is less than the slope of E itself.

The generalization of Narasimhan – Seshadri uniqueness theorem for stable Higgs pairs states that [17, 24] if (E, θ) is stable and $c_1(E) = 0$ there is a unique unitary connection ∇_H compatible with the holomorphic structure, such that

$$F_H + [\theta, \theta^*] = 0 \quad (3.1)$$

The notion of Higgs system can be related to the one of *harmonic bundle*. Let V be a complex rank n vector bundle equipped with a flat connection ∇ . The introduction of an hermitian fiber metric H on V amounts to a reduction of the structure group to the unitary subgroup, and allows for a splitting of the connection as

$$\nabla = \nabla_H + \alpha \quad (3.2)$$

where ∇_H is a unitary connection and α is a 1-form with values in the self-adjoint part of $\text{End}(V)$. The zero-curvature equations for the connection ∇ are [5]

$$\nabla_H^2 + \frac{1}{2}[\alpha, \alpha] = 0, \quad \nabla_H \alpha = 0 \quad (3.3)$$

The pair (V, ∇) is said [5, 25, 8] to be *harmonic* if we have in addition

$$\nabla_H^* \alpha = 0 \quad (3.4)$$

where the adjoint is taken with respect to a given metric on Σ .

Since α is self-adjoint, we can decompose it as $\alpha = \theta + \theta^*$, thus showing that equations (3.3, 3.4) are equivalent to Hitchin's self-duality equation (3.1), supplemented by $\bar{\partial}\theta = 0$, with the $\bar{\partial}$ -operator coming from the $(0,1)$ part of ∇_H .

Recall that a metric H can be considered as a multi-valued mapping $f_H : \Sigma \rightarrow GL(n, \mathbb{C})/U(n)$ or, in other words, as a section of a bundle over Σ whose standard fiber is the coset $GL(n, \mathbb{C})/U(n)$. Since ∇ is flat, the section f_H can be regarded as a map

from the universal cover of Σ , $f_H : \tilde{\Sigma} \rightarrow GL(n, \mathbb{C})/U(n)$, equivariant with respect to the action of $\pi_1(\Sigma)$, i.e., we have the commutative diagram

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{f_H} & GL(n, \mathbb{C})/U(n) \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{f_H} & \Gamma \backslash GL(n, \mathbb{C})/U(n) \end{array}$$

where $\pi_1(\Sigma)$ acts on $\tilde{\Sigma}$ as the group of deck transformations and on $GL(n, \mathbb{C})$ via the holonomy representation Γ . In the flat coordinate system for V in which $\nabla \equiv d$ one has

$$\alpha = -\frac{1}{2} f_H^{-1} df_H \quad (3.5)$$

which means that α can be identified with the differential of f_H . Equation (3.4) implies that the map f_H is harmonic [9] and explains the terminology.

The link between the Toda equations and Higgs bundles can be given as follows. Let \mathfrak{g} be a simple finite dimensional Lie algebra. A Toda field over a Riemann surface Σ is a field Φ taking values in the Cartan subalgebra \mathfrak{k} of \mathfrak{g} and the Toda equations

$$\partial_z \partial_{\bar{z}} \Phi = \sum h_i e^{\alpha_i(\Phi)} \quad (3.6)$$

can be obtained as the zero curvature equations [20] for the connection

$$A_z = \frac{1}{2} \partial_z \Phi + \exp\left(\frac{1}{2} \text{ad} \Phi\right) \cdot \mathcal{E}_+ \quad (3.7)$$

$$A_{\bar{z}} = -\frac{1}{2} \partial_{\bar{z}} \Phi + \exp\left(-\frac{1}{2} \text{ad} \Phi\right) \cdot \mathcal{E}_- \quad (3.8)$$

where \mathcal{E}_+ (\mathcal{E}_-) denote the sum of the positive (negative) simple roots.

Let us now consider the gauge transformed connection under the element $g = \exp(\frac{1}{2} \Phi)$ [3]. We have

$$A_z^g = \partial_z \Phi + \mathcal{E}_+ \quad A_{\bar{z}}^g = \exp(-\text{ad} \Phi) \cdot \mathcal{E}_- \quad (3.9)$$

We can now consider $\exp(\Phi)$ as an hermitian form on the fibers, split $D_A = d + A$ as

$$D_H + \theta + \tilde{\theta} \quad (3.10)$$

where D_H is the metric connection associated to $H = \exp(\Phi)$,

$$\theta = \mathcal{E}_+ dz \quad (3.11)$$

and $\tilde{\theta} = H^{-1} \mathcal{E}_- H d\bar{z}$. Namely we have that

$$\begin{aligned} D_H' &= \partial + \partial \Phi \\ D_H'' &= \bar{\partial} \end{aligned}$$

Notice that $\tilde{\theta}$ is the metric adjoint endomorphism of θ . Hence, together with the obvious fact that $D_H''\theta = 0$, the zero-curvature equations in this gauge are

$$D_H^2 + [\theta, \theta^*] = 0 \quad (3.12)$$

thus showing that any solution to the Toda equations gives rise to a well defined solution of the Hitchin's equations for a suitable Higgs pair. In particular, if $\mathfrak{g}^{\mathbb{C}} = A_{n-1}$, the underlying vector bundle E is

$$E = \bigoplus_{r=0}^{n-1} K^{-\frac{n-1}{2}+r} \quad (3.13)$$

The metric H is given by a diagonal matrix whose entries $h_r = e^{\varphi_r}$, $r = 1, \dots, n$, are themselves metrics on the factors $K^{-\frac{n-1}{2}+r-1}$ appearing in (3.13). This completely fixes the transformation law of the fields φ_r , (which are related to the “true” Toda fields ϕ_i by a standard overparametrization) and one can check that it coincides with the well-known conformal transformation properties of the Toda fields [3].

4 Variations of Hodge Structures

The purpose of this section is to bring to the light how solutions of the Toda equations determine local *holomorphic* maps from Σ to a symmetric hermitian manifold. These are to be considered as the generalization of the uniformizing maps associated to local solutions of the Liouville equation briefly recalled in section (2). The crucial properties for this analysis are the natural filtration of the Higgs bundle, as well as the existence of *two* real structures, which enable one to make full use of the theory of Variations of Hodge structures along the lines of [25].

4.1 The real structures

Consider the basic Higgs system given by (3.11) and (3.13) together with the harmonic metric H . The natural extension of Serre duality defines a symmetric bilinear map $S : E \otimes E \rightarrow \mathcal{O}_{\Sigma}$ satisfying

$$S(\theta u, v) = S(u, \theta v)$$

for any two local sections u, v of E . This can be used to ensure that in certain representations the structure group, and hence the holonomy, is reduced to a real form of $G^{\mathbb{C}} = SL(n, \mathbb{C})$. Actually, as found by Hitchin [18] it is the adjoint group of the *split* form $G^r = SL(n, \mathbb{R})$. It is perhaps of some interest to notice that the conjugation τ in $sl(n, \mathbb{C})$ which selects the real form is concretely given by

$$\tau(\xi) = S \bar{\xi} S, \quad \xi \in sl(n, \mathbb{C}) \quad (4.1)$$

where S is the matrix

$$S = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix} \quad (4.2)$$

We now show there exists another real structure on the vector bundle E . Let $A : E \rightarrow E$ be the endomorphism equal to $(-1)^r$ on each factor $K^{-\frac{n-1}{2}+r}$. With it, we construct an indefinite hermitian form $\langle \cdot, \cdot \rangle$ over E , namely

$$\langle u, v \rangle = (A u, v)_H, \quad u, v \in E \quad (4.3)$$

A straightforward calculation proves that the hermitian form (4.3) is flat with respect to the Toda connection $D = D_H + \theta + \theta^*$, that is we have

$$d \langle u, v \rangle = \langle D u, v \rangle + \langle u, D v \rangle \quad u, v \in E$$

This implies, of course, that a reduction of the structure group from $SL(n, \mathbb{C})$ to $SU(p, q)$, where $p = [n/2]$, $q = n - p$, takes place. More precisely, what we actually mean by “ $SU(p, q)$ ” is the group corresponding to the fixed point set in $\mathfrak{g}^{\mathbb{C}} = A_{n-1}$ of the conjugation ν given by

$$\nu(\xi) = -I \rho(\xi) I, \quad \xi \in \mathfrak{g}^{\mathbb{C}} \quad (4.4)$$

where in this case ρ is simply minus the hermitian conjugate and I is the matrix having alternatively ± 1 on the principal diagonal. The conjugations τ defined in (4.1) and ν commute, so that, upon carefully choosing the representation, (the Lie algebra of) the structure group of the harmonic bundle corresponding to the Toda equations can be reduced to the intersection of the fixed point sets of τ and ν . Let us call G the real structure group so obtained and K its maximal compact subgroup.

By the results about harmonic bundles quoted in section (3), we thus obtain a harmonic map $f_H : \Sigma \rightarrow \Gamma \backslash G/K$ and we can interpret the Toda field equations as the equations characterizing the embedding of the Riemann surface into a some homogeneous manifold through a harmonic map f_H . We can actually refine this, that is starting from the map f_H we can define a holomorphic embedding $F : \Sigma \rightarrow \mathbb{D}$ into a *complex* manifold \mathbb{D} , fibered over G/K . This requires a more extensive analysis of the structure of the bundle we associated to the Toda equations.

4.2 Toda systems and Variations of Hodge structures

Upon rewriting our rank- n basic bundle (3.13) as

$$E = \bigoplus_{r+s=n-1} E^{r,s}, \quad E^{r,s} = K^{-\frac{n-1}{2}+r} \quad (4.5)$$

the Higgs field θ appearing in the Toda connection, eq. (3.11), has the property

$$\theta : E^{r,s} \longrightarrow E^{r-1,s+1} \otimes K \quad (4.6)$$

and the factors are orthogonal with respect to both the metric H and the indefinite hermitian form $\langle \cdot, \cdot \rangle$. As a consequence, the complete connection $D = D_H + \theta + \theta^*$ satisfies the following *Griffiths transversality condition*

$$D : E^{r,s} \longrightarrow A^{1,0}(E^{r-1,s+1}) \oplus A^{1,0}(E^{r,s}) \oplus A^{0,1}(E^{r,s}) \oplus A^{0,1}(E^{r+1,s-1}) \quad (4.7)$$

where by $A^\bullet(E^{r,s})$ we mean C^∞ sections. It is useful for later purposes to rewrite (4.7) in the following form. Consider the filtration

$$E \equiv F^0 \supset F^1 \supset \dots \supset F^{n-1} \supset F^n \equiv \{0\} \quad (4.8)$$

where

$$F^q = \bigoplus_{r=q}^{n-1} K^{-\frac{n-1}{2}+r} = \bigoplus_{r=q}^{n-1} E^{r,s} \quad (4.9)$$

Then the transversality condition can be restated as

$$\begin{aligned} D' : F^q &\longrightarrow A^{1,0}(F^{q-1}) \\ D'' : F^q &\longrightarrow A^{0,1}(F^q) \end{aligned} \quad (4.10)$$

According to Simpson, a harmonic bundle $E = \bigoplus_{r+s=w} E^{r,s}$ whose factors are orthogonal with respect to an indefinite hermitian form $\langle \cdot, \cdot \rangle$, satisfying ((4.6)) defines a *complex variation of Hodge structure* [24, 25, 15]

It is outside of the scope of this paper to give a complete introduction to the theory of variations of Hodge structures; however its basics are the following.

Let us denote by \mathbf{E} a complex vector space equipped with

- a conjugation $\cdot^\sigma : \mathbf{E} \rightarrow \mathbf{E}$
- a bilinear form $Q : \mathbf{E} \times \mathbf{E} \rightarrow \mathbb{C}$ such that:
 1. $Q(v, u) = (-1)^w Q(u, v)$, $u, v \in \mathbf{E}$,
 2. it is “real” with respect to the conjugation of \mathbf{E} , namely $\overline{Q(u, v)} = Q(u^\sigma, v^\sigma)$, $u, v \in \mathbf{E}$.

A period domain \mathbb{D} (or, in words, the set of all weight w Hodge structures on \mathbf{E}) can be defined to be the set of all (descending) filtrations $\{\mathbf{F}^q\}$ in \mathbf{E} such that

$$\begin{aligned} Q(\mathbf{F}^q, \mathbf{F}^{w-q+1}) &= 0 \\ Q(Cu, u^\sigma) &> 0 \end{aligned} \quad (4.11)$$

where C restricts to $\sqrt{-1}^{r-s}$ on each of the quotients $E^{r,s} = F^r / F^{r+1}$.

Dropping the second condition in the definition, yields the *compact dual* $\check{\mathbb{D}}$ of \mathbb{D} . It is an algebraic subvariety (actually a *manifold*) of a flag manifold, and hence of a

product of Grassmannians [15], in which the period domain \mathbb{D} lies as an open subset, and therefore as a complex submanifold.

The Higgs bundle associated to the Toda equations displays the formal properties of a Variation of Hodge Structure of weight $w = n - 1$ whose subsequent quotients $E^{r,s} = K^{-\frac{n-1}{2}+r} \cong F^r/F^{r+1}$ are line bundles. Namely on each fibre we have the conjugation \cdot^σ as $u^\sigma = S\bar{u}$ (S is given by (4.2)), and the bilinear form Q is given by (4.3). Applying Griffiths' theory of Variations of Hodge Structure, and essentially the transversality condition (4.7) one proves [2]

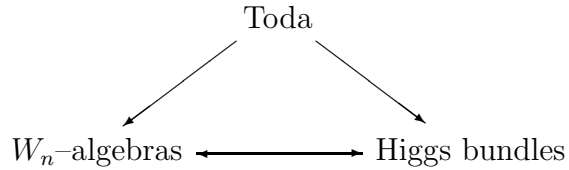
Theorem 4.1 *The Toda equations determine a holomorphic embedding*

$$F_H : \Sigma \longrightarrow \Gamma \backslash \mathbb{D}$$

where Γ is the monodromy group, $\mathbb{D} \cong G/K_0$ a Griffiths period domain, G is the structure group defined in §5.1 and $K_0 \subset K \subset G$ a (compact) subgroup. The map F_H is the metric H seen as a section of a flat bundle over Σ with typical fiber G/K_0 and its differential is given by the Higgs field θ .

5 Toda fields and W_n -geometry

So far we have analyzed the correspondence between Toda field theory and Higgs bundles in the framework of the theory of hermitian holomorphic vector bundles on a generic genus Riemann surface Σ . We now come to the last issue, i.e. “closing” the triangular correspondence



From the point of view of the theory of connections on higher genus Riemann surfaces the relations between Toda Field theory and W_n -algebras is better understood as follows. The $(0,1)$ part of any connection ∇ on Σ is integrable by dimensional reasons, thus giving a holomorphic structure to the complex vector bundle supporting it. In this holomorphic frame one has $\nabla'' = \bar{\partial}$.

The characterization of the holomorphic bundle associated to the basic Higgs bundle (3.13) and equipped with the connection (the *Toda*-connection)

$$D' = \partial + \begin{pmatrix} \partial\varphi_1 & 1 & & \\ & \partial\varphi_2 & 1 & \\ & & \ddots & \\ & & & 1 \\ & & & & \partial\varphi_n \end{pmatrix} \quad D'' = \bar{\partial} + \begin{pmatrix} 0 & & & & \\ e^{\varphi_1 - \varphi_2} & 0 & & & \\ & e^{\varphi_2 - \varphi_3} & 0 & & \\ & & \ddots & \ddots & \\ & & & e^{\varphi_{n-1} - \varphi_n} & 0 \end{pmatrix} \quad (5.1)$$

(here $\sum_{i=1}^n \varphi_i = 0$), is given [2] by the following

Theorem 5.1 *The holomorphic vector bundle V defined by the flat Toda connection $D = D_H + \theta + \theta^*$ is the vector bundle of $(n-1)$ -jets of sections of $K^{-\frac{n-1}{2}}$. The holomorphic connection ∇ , which is the image of the Toda connection D has the standard W (or Drinfel'd-Sokolov) form:*

$$\nabla' = \partial + \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 0 & 1 \\ w_n & w_{n-1} & \cdots & w_2 & 0 \end{pmatrix}, \quad \nabla'' = \bar{\partial} \quad (5.2)$$

with $\bar{\partial} w_i = 0$, $i = 2, \dots, n$.

The proof of this theorem boils down to show that the vector bundle E , associated in a suitable covering $\{\mathcal{U}_\alpha\}$ of Σ by the $SL(n, \mathbb{C})$ -cocycle

$$\mathcal{E}_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^{\frac{n-1}{2}} & & & & \\ & k_{\alpha\beta}^{\frac{n-1}{2}-1} & & & \\ & & \ddots & & \\ & & & k_{\alpha\beta}^{-\frac{n-1}{2}+1} & \\ & & & & k_{\alpha\beta}^{-\frac{n-1}{2}} \end{pmatrix} \quad (5.3)$$

equipped with the connection D is C^∞ -equivalent to the bundle V of $(n-1)$ -jets of sections of $K^{-\frac{n-1}{2}}$, equipped with the connection ∇ . We recall that the transition functions $\mathcal{V}_{\alpha\beta}$ of V can be gotten by expanding the relation $\partial_\alpha^l \xi_\alpha = (k_{\alpha\beta}^{-1} \partial_\beta)^l (k_{\alpha\beta}^{\frac{n-1}{2}} \xi_\beta)$, ξ_α being a local section of $K^{-\frac{n-1}{2}}$, and $k_{\alpha\beta} = \frac{\partial z_\alpha}{\partial z_\beta}$.

Rather than dwell at large on the proof, consistent parts of which were already known or implicit in the literature, we will examine in some details the A_2 case.

The transition functions for the 2-jet bundle of K^{-1} are given by

$$\begin{pmatrix} \sigma_\alpha \\ \partial_\alpha \sigma_\alpha \\ \partial_\alpha^2 \sigma_\alpha \end{pmatrix} = \begin{pmatrix} k_{\alpha\beta} & 0 & 0 \\ \partial_\beta \log k_{\alpha\beta} & 1 & 0 \\ k_{\alpha\beta}^{-2} \partial_\beta^2 \log k_{\alpha\beta} & k_{\alpha\beta}^{-1} \partial_\beta \log k_{\alpha\beta} & k_{\alpha\beta}^{-1} \end{pmatrix} \begin{pmatrix} \sigma_\beta \\ \partial_\beta \sigma_\beta \\ \partial_\beta^2 \sigma_\beta \end{pmatrix} \quad (5.4)$$

The isomorphism (in the C^∞ -category) between $V = J^2(K^{-1})$ and $E = K^{-1} \oplus \mathbb{C} \oplus K$ is accomplished by a smooth $SL(3, \mathbb{C})$ -valued 0-cochain G_α which we find in the factorized form $G_\alpha = G_\alpha^{(1)} G_\alpha^{(2)}$, with

$$G_\alpha^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ h_\alpha & 1 & 0 \\ f_\alpha & 0 & 1 \end{pmatrix} \quad G_\alpha^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g_\alpha & 1 \end{pmatrix}$$

The Toda connection looks like

$$A_z = \begin{pmatrix} \partial\varphi_1 & 1 & 0 \\ 0 & \partial\varphi_2 & 1 \\ 0 & 0 & \partial\varphi_1 \end{pmatrix} \quad A_{\bar{z}} = \begin{pmatrix} 0 & 0 & 0 \\ e^{\varphi_1 - \varphi_2} & 0 & 0 \\ 0 & e^{\varphi_2 - \varphi_3} & 0 \end{pmatrix} \quad (5.5)$$

Provided we set $h_\alpha = -(\partial\varphi_1)_\alpha$, $g_\alpha = (\partial\varphi_3)_\alpha$, under the transformation G_α the co-cycle (5.4) is sent into its diagonal part and the Toda connection into its Drinfel'd – Sokolov partner

$$(A')^G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w_3 & w_2 & 0 \end{pmatrix}, \quad (A'')^G = 0 \quad (5.6)$$

Such a procedure yields the usual representation [4] for the generators of the W_3 -algebra:

$$w_2 = (\partial\phi_1)^2 + (\partial\phi_2)^2 - [\partial^2\phi_1 + \partial^2\phi_2 + \partial\phi_1\partial\phi_2] \quad (5.7)$$

and

$$w_3 = \partial w_2 + (\partial\phi_1)^2\partial\phi_2 - \partial\phi_1(\partial\phi_2)^2 + 2\partial\phi_2\partial^2\phi_2 - \partial^3\phi_2. \quad (5.8)$$

Connecting this picture with the results of the previous sections, we see how datum of a A_{n-1} -Toda Field on Σ (and so the datum of a realization of the W_n -algebra) allows to regard the Riemann Surface Σ as a base space for a Variation of Hodge Structure, and henceforth yields a holomorphic map from Σ into a quotient of a Griffiths period domain G/K_0 .

These results are clearly linked to the so called W_n -embeddings as discussed by Gervais, Saveliev and collaborators, in their works on the geometrical meaning of the extended local symmetries. and the eventual characterization of the W_n -moduli spaces. Although the task is clearly much more difficult than in the Virasoro case, because of the non-linearity of W_n -algebras, the correct geometrical backgrounds were pointed out and important steps forwards were made in [26, 12, 14], where the extrinsic geometry of “chiral” embeddings of Σ in some projective or affine space were studied.

In particular, in the paper [23] the following picture is explained. The starting point is a smooth map from Σ to a complex Lie group G . Whenever it satisfies the “grading condition”, it induces a holomorphic map $\varphi_P : \Sigma \rightarrow G/P$, P being a parabolic subgroup of G . Considering those parabolic subgroups $P_i, i = 1, \dots, \text{rank } G$, for which G/P_i is the i^{th} fundamental homogeneous space for G , the associated maps φ_{P_i} define maps from Σ to $\mathbb{P}(V_i)$, the projectivization of the i^{th} fundamental representation of G . Then it is shown that the (generalized) Plücker relations for the curvature of the pull-back on Σ of the Fubini–Study metrics on $\mathbb{P}(V_i)$ on Σ translate, when expressed through local Kähler potentials, into the Toda Field equations for a suitably chosen local representative of φ_{P_i} .

Our approach can be considered as a sort of inverse path: we *start* from a solution of the Toda Field equations and we *determine* a holomorphic map from Σ to a suitable locally homogeneous space. It follows that the target space we obtain is only *locally*

determined by the rank of the Cartan subalgebra in which the Toda fields take values, since in the large the monodromy action of $\pi_1(\Sigma)$ on the Griffiths period domain must be factored out, thus yielding a different global target space according to the genus $g(\Sigma)$.

Nonetheless, Plücker formulas are of local type, so one should expect them to arise also in our context. Indeed, one can argue as follows. Let us overparametrize the Toda fields ϕ_i by

$$\varphi_{-\frac{n-1}{2}+r} = \phi_{r+1} - \phi_r, \quad \phi_0 = \phi_n = 0 \quad (5.9)$$

which amounts to a renumbering of the fields φ_i entering the Toda connection. Looking at the filtration (4.8) one sees that the metric on F_q is the rank $n - q$ matrix

$$H_q = H|_{F^q} = \begin{pmatrix} e^{\varphi_{-\frac{n-1}{2}+r}} & & \\ & \ddots & \\ & & e^{\varphi_{\frac{n-1}{2}}} \end{pmatrix} \quad (5.10)$$

so that $\det H_q = \exp(\sum_{r=q}^{n-1} \varphi_{-\frac{n-1}{2}+r})$ is a metric on $\wedge^{\max} F^q \equiv \det F^q$. The defining relations (5.9) imply that $\log \det H_q = -\phi_q$, therefore the metric connection on $\det F^q$ is $-\partial\phi_q$, and its associated curvature is $\bar{\partial}\partial\phi_q$. Writing explicitly the Toda equations as

$$\bar{\partial}\partial\phi_q = \exp(2\phi_q - \phi_{q-1} - \phi_{q+1}), \quad q = 1, \dots, n \quad (5.11)$$

we see that the left hand side is the curvature of $\det F^q$, and the right hand side is a metric on the line bundle $(\det F^q)^{-2} \det F^{q-1} \det F^{q+1}$.

Recalling that $F^q/F^{q-1} = K^{-(n-1)/2+q} \equiv \det F^q (\det F^{q-1})^{-1}$ we get that $\exp(2\phi_q - \phi_{q-1} - \phi_{q+1})$ is a metric g_q on K^{-1} , i.e. a metric on Σ , and then the Toda equations tell us that ϕ_q is the Kähler potential for g_q , whose associated 2-form is $\omega_q = (\sqrt{-1}/2) \partial\bar{\partial}\phi_q$. The infinitesimal Kähler relations

$$\text{Ric}_r = \omega_{r-1} + \omega_{r+1} - 2\omega_r \quad (5.12)$$

follow from standard definitions.

To recover the relations in terms of Plücker coordinates, one can take advantage of the naturality of the above local constructions with respect to the natural embeddings of the Griffiths' domain $\check{\mathbb{D}}$ into the product of Grassmannians [15]

$$G(h_1, n) \times G(h_2, n) \cdots \times G(h_{n-1}, n)$$

(h_r is the rank of F^r in the Hodge filtration), namely with respect to the pull backs to $\check{\mathbb{D}}$ of the determinant line bundles associated to the tautological sequences

$$0 \longrightarrow S_{h_r} \longrightarrow \mathbb{C}^n \longrightarrow Q_{h_r} \longrightarrow 0$$

over $G(h_r, n)$.

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